Asymptotic behavior of the unbounded solutions of some boundary layer equations

by

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Abstract. We give an asymptotic equivalent at infinity of the unbounded solutions of some boundary layer equations arising in fluid mechanics.

Let us consider the following boundary layer differential equation

$$f''' + ff'' - \beta f'^2 = 0 \tag{1}$$

where $\beta < 0$. We are interested in non constant solutions (that we will simply call *solutions*) of (1) defined on some interval $[t_0, \infty)$ and such that

$$f'(\infty) := \lim_{t\to\infty} f'(t) = 0. \tag{2}$$

Equation (1) can be obtained from similarity boundary layer equations as those introduced by numerous authors in [1], [2], [11], [12], [13], [14], [17] and [18], and studied from mathematical point of view in [3], [4], [6], [7], [9], [10] and [15]. In these papers, the corresponding differential equation is considered on $[0, \infty)$ with the boundary conditions $f(0) = a$, $f'(0) = 1$ and (2), or $f(0) = a$, $f''(0) = -1$ and (2). Here, we will be concerned by unbounded solutions of these problems, and to be as general as possible we will consider all the unbounded solutions of (1)-(2) defined on some interval $[t_0, \infty)$. The restriction to $\beta < 0$ is due to the fact that for $\beta \geq 0$ none of the solutions of (1)-(2) are unbounded (see Remark 6 below).

For $\beta = 0$, equation (1) reduces to the Blasius equation and a lot of papers have been published about it. To have a survey, we refer to [16], [5], [8] and the references therein.

Concerning the existence of unbounded solutions of (1)-(2), elementary direct methods give it for $-2 \leq \beta < 0$ (see for example [7] and [15]). It seems more difficult to get such existence results for $\beta < -2$ and the best way to overcome this difficulty should consist in introducing appropriate blow-up coordinates. Precisely, if f is a solution of (1) which does not vanish on some interval I, we set

$$\forall t \in I, \qquad s = \int_\tau^t f(\xi)d\xi, \qquad u(s) = \frac{f'(t)}{f(t)^2} \qquad \text{and} \qquad v(s) = \frac{f''(t)}{f(t)^3}.$$

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Then, we easily get

$$\begin{cases} \dot{u} = v - 2u^2, \\ \dot{v} = -v + \beta u^2 - 3uv, \end{cases}$$

where the dot is for differentiating with respect to the variable s. The plane dynamical system that we obtain has the origin as a saddle-node, and studying the phase portrait in the neighbourhood of it allows us to underscore the fact that unbounded positive or negative solutions of (1)-(2) have to exist. For details, see [9] or [10].

We now focus our attention on the behavior at infinity of these unbounded solutions. We start by some elementary and useful lemmas.

Lemma 1. *Let f be a solution of* (1) *defined on some interval J. If there is $\tau \in J$ such that $f''(\tau) \leq 0$, then for all $t \in J$ such that $t > \tau$ we have $f''(t) < 0$.*

`Proof.` This follows immediately from the equality $(f''e^F)' = \beta f'^2 e^F$, where F denotes any anti-derivative of f on J, and from the fact that f' and f'' cannot vanish together without f being constant. □

Lemma 2. *Let f be a solution of* (1)-(2) *defined on some interval $[t_0, \infty)$. There exists $t_1 \geq t_0$ such that $f''(t)f'(t) < 0$ and $f'''(t)f''(t) < 0$ for $t \geq t_1$.*

`Proof.` By Lemma 1, we know that f'' cannot vanish more than once on $[t_0, \infty)$ and thus there exists $t_2 \geq t_0$ such that

$$\forall t > t_2, \qquad f''(t)f'(t) = -f''(t) \int_t^\infty f''(s)ds < 0.$$

Differentiating (1) we get $f^{(iv)} + ff''' - (2\beta - 1)f'f'' = 0$ and $(f'''e^F)' = (2\beta - 1)f''f'e^F$ where F denotes any anti-derivative of f on $[t_2, \infty)$. It follows that f''' cannot vanish more than once on $[t_2, \infty)$ in such a way that $f''(t) \to 0$ as $t \to \infty$ and there exists $t_1 \geq t_2$ such that

$$\forall t \geq t_1, \qquad f'''(t)f''(t) = -f'''(t) \int_t^\infty f'''(s)ds < 0.$$

This completes the proof. □

We now are able to prove our main result.

Theorem 3. *Let f be an unbounded solution of* (1)-(2). *There exists a constant $c > 0$ such that*

$$|f(t)| \sim ct^{\frac{1}{1-\beta}} \qquad as \qquad t \to \infty. \tag{3}$$

`Proof.` Let $f : [t_0, \infty) \to \mathbb{R}$ be an unbounded solution of (1)-(2).

Case 1. Let us assume first that f is positive at infinity. Thanks to Lemma 2, there exists $t_1 \geq t_0$ such that

$$\forall t \geq t_1, \qquad f(t) > 0, \quad f'(t) > 0, \quad f''(t) < 0 \quad \text{and} \quad f'''(t) > 0.$$

Therefore, on (t_1, ∞), we have $(f'f^{-\beta})' = (ff'' - \beta f'^2)f^{-\beta-1} = -f'''f^{-\beta-1} < 0$ in such a way that the function $\phi = f'f^{-\beta}$ is decreasing on $[t_1, \infty)$ and

$$\phi(t) = f'(t)f^{-\beta}(t) \longrightarrow l_0 \in [0, \infty) \quad \text{as} \quad t \to \infty. \tag{4}$$

Now, multiplying equation (1) by $f^{-\beta-1}$ and integrating between $s \geq t_1$ and $t \geq s$ we easily get

$$f^{-\beta-1}(t)f''(t) - f^{-\beta-1}(s)f''(s) + f^{-\beta}(t)f'(t) - f^{-\beta}(s)f'(s)$$
$$= -(\beta+1)\int_s^t f^{-\beta-2}(r)f'(r)f''(r)dr. \tag{5}$$

Since $ff'f'' < 0$ on (t_1, ∞), the right hand side of (5) has a limit as $t \to \infty$ and thus from (4) we deduce that $f^{-\beta-1}(t)f''(t)$ has a limit $l_1 \in [-\infty, 0]$ as $t \to \infty$. Suppose now $l_1 < 0$. Then there exist $l_2 \in (l_1, 0)$ and $t_2 > t_1$ such that $f^{-\beta-1}(t)f''(t) < l_2$ for $t > t_2$. It follows that

$$\forall t > t_2, \qquad f''(t) < l_2 f^{\beta+1}(t) < \frac{l_2}{\phi(t_1)} f'(t)f(t).$$

Integrating, we get

$$\forall t > t_2, \qquad f'(t) - f'(t_2) < \frac{l_2}{2\phi(t_1)}(f^2(t) - f^2(t_2))$$

and a contradiction with (2) since the right hand side tends to $-\infty$ as $t \to \infty$. Consequently $l_1 = 0$ and coming back to (5) we get

$$l_0 = f^{-\beta-1}(s)f''(s) + f^{-\beta}(s)f'(s) - (\beta+1)\int_s^\infty f^{-\beta-2}(r)f'(r)f''(r)dr, \tag{6}$$

and this equality holds for all $s \geq t_1$. It remains to show that $l_0 > 0$. For that we have to distinguish between the cases $\beta \geq -1$ and $\beta < -1$.

Assume first that $\beta \geq -1$. Then (6) implies that

$$l_0 \geq \sup_{s \geq t_1}\{f^{-\beta-1}(s)f''(s) + f^{-\beta}(s)f'(s)\} > 0$$

because, on the contrary, we should have $f''(s) + f(s)f'(s) \leq 0$ for all $s \geq t_1$, and by integrating

$$\forall s \geq t_1, \qquad f'(s) + \frac{1}{2}f^2(s) \leq f'(t_1) + \frac{1}{2}f^2(t_1)$$

which is absurd since $f(s) \to \infty$ as $s \to \infty$.

Assume now that $\beta < -1$. Since the function ϕ is decreasing, we have

$$\int_s^\infty f^{-\beta-2}(r)f'(r)f''(r)dr \geq f^{-\beta-2}(s)f'(s)\int_s^\infty f''(r)dr = -f^{-\beta-2}(s)f'(s)^2.$$

We then deduce from (6) that

$$\forall s \geq t_1, \qquad l_0 \geq f^{-\beta-2}(s)\{f'(s)f^2(s) + f''(s)f(s) + (\beta+1)f'(s)^2\}.$$

Looking next at the polynomial $P_s(X) = f'(s)X^2 + f''(s)X + (\beta+1)f'(s)^2$, we easily see that for

$$X > -\frac{f''(s)}{f'(s)} + \sqrt{-(\beta+1)f'(s)},$$

we have $P_s(X) > 0$. To conclude, it is sufficient to remark that there exists $s_0 \geq t_1$ such that

$$f(s_0) > -\frac{f''(s_0)}{f'(s_0)} + \sqrt{-(\beta+1)f'(s_0)}. \tag{7}$$

Indeed, on the contrary we should have $f'(s) \to \infty$ as $s \to \infty$ and a contradiction. Therefore (7) holds and we have $l_0 \geq f^{-\beta-2}(s_0)P_{s_0}(f(s_0)) > 0$.

Finally, we have $f'(t)f(t)^{-\beta} \sim l_0$ as $t \to \infty$, and by integrating we obtain

$$f(t)^{-\beta+1} \sim l_0(1-\beta)t \quad \text{as} \quad t \to \infty$$

and the result in this case.

<u>Case 2</u>. Let us assume now that f is negative at infinity. Thanks to Lemma 2, there exists $t_1 \geq t_0$ such that

$$\forall t \geq t_1, \qquad f(t) < 0, \quad f'(t) < 0, \quad f''(t) > 0 \quad \text{and} \quad f'''(t) < 0.$$

Then, on $[t_1, \infty)$, we have $(f'(-f)^{-\beta})' = (-ff'' + \beta f'^2)(-f)^{-\beta-1} = f'''(-f)^{-\beta-1} < 0$ in such a way that the function $\psi = f'(-f)^{-\beta}$ is decreasing and we have

$$\psi(t) = f'(t)(-f(t))^{-\beta} \longrightarrow l_0 \in [-\infty, 0) \quad \text{as} \quad t \to \infty. \tag{8}$$

To conclude as in the first case, it is sufficient to prove that l_0 is finite. Multiplying equation (1) by $(-f)^{-\beta-1}$ and integrating between $s \geq t_1$ and $t \geq s$ we easily get

$$f''(t)(-f(t))^{-\beta-1} - f''(s)(-f(s))^{-\beta-1} - f'(t)(-f(t))^{-\beta} + f'(s)(-f(s))^{-\beta}$$
$$= (\beta+1)\int_s^t f'(r)f''(r)(-f(r))^{-\beta-2}dr. \tag{9}$$

If $\beta \geq -1$, then the right hand side of (9) is non positive, and we see immediatly that l_0 has to be finite. Let us assume now that $\beta < -1$ and choose s such that $f'(s)f(s)^{-2} > \frac{1}{2(\beta+1)}$. Since the function ψ is decreasing and negative, we get

$$\int_s^t f'(r)f''(r)(-f(r))^{-\beta-2}dr \geq \psi(t)f(s)^{-2}\int_s^t f''(r)dr \geq -\frac{1}{2(\beta+1)}\psi(t).$$

Then, setting $C(s) = f''(s)(-f(s))^{-\beta-1} - f'(s)(-f(s))^{-\beta}$, we easily deduce from (9) that

$$f''(t)(-f(t))^{-\beta-1} - \psi(t) - C(s) \leq -\frac{1}{2}\psi(t),$$

which gives $\psi(t) \geq -2C(s)$. Thus l_0 is finite. □

Remark 4. If $\beta = -1$, then $|l_0| = f''(t_1) + f(t_1)f'(t_1)$ and $|f(t)| \sim \sqrt{2|l_0|t}$ as $t \to \infty$.

Remark 5. For any $\beta \in \mathbb{R}$ and any $\tau \in \mathbb{R}$, the function

$$t \longmapsto \frac{6}{(2-\beta)(t-\tau)}$$

is a bounded convex solution of (1)-(2) on $[t_0, \infty)$ for all $t_0 > \tau$. Bounded concave solutions of (1)-(2) exist too (see [4], [5], [6], [7], [9] and [10]).

Remark 6. For $\beta \geq 0$, the solutions of (1)-(2) are always bounded. In fact, suppose that $f : [t_0, \infty) \to \mathbb{R}$ is a solution of (1)-(2), then we have $f''' \geq -ff''$ in such a way that if $f < 0$ at infinity, we deduce from Lemma 2 that there exists $t_1 \geq t_0$ such that necessarily $f'' < 0$ and $f''' > 0$ on $[t_1, \infty)$. Such a f is bounded.

If now $f > 0$ at infinity and is unbounded, then $f(t) \to \infty$ as $t \to \infty$ and there exists $t_1 \geq t_0$ such that $f'' < 0$ and $f > 1$ on $[t_1, \infty)$. Therefore $f'''(t) \geq -f''(t)$ for $t \geq t_1$, and by integrating between $s \geq t_1$ and ∞ we obtain $-f''(s) \geq f'(s)$. Integrating next between t_1 and $t \geq t_1$, we get $f'(t_1) - f'(t) \geq f(t) - f(t_1)$ and a contradiction by passing to the limit as $t \to \infty$.

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